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INTEGRATION OF ELLIPTIC INTEGRALS.

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[Continued from December Number.]

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta\cos^4\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}=\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^4\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}-\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^6\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}.$$

∴ From (12) and (19) by substitution we get,

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^6\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}=\frac{1}{105e^8}[(48-128e^2+103e^4-15e^6)E(e,\ \tfrac{1}{2}\pi)$$

$$-4(12-38e^2+41e^4-15e^6)F(e,\ \tfrac{1}{2}\pi)]\ldots\ldots(20).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^6\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}=\int_0^{\frac{1}{2}\pi}\frac{\cos^6\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}-\int_0^{\frac{1}{2}\pi}\frac{\cos^8\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}.$$

∴ From (13) and (20) by substitution we get,

$$\int_0^{\frac{1}{2}\pi}\frac{\cos^8\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}=\frac{1}{105e^8}[(48-208e^2+353e^4-298e^6+105e^8)F(e,\ \tfrac{1}{2}\pi)$$

$$-2(24-92e^2+132e^4-88e^6)E(e,\ \tfrac{1}{2}\pi)]\ \ldots\ (21).$$

$$\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}\sin^6\theta d\theta=\int_0^{\frac{1}{2}\pi}\frac{\sin^6\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}-e^2\int_0^{\frac{1}{2}\pi}\frac{\sin^8\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}.$$

∴ From (10) and (17) by substitution we get,

$$\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}\sin^6\theta d\theta=\frac{1}{105e^6}[(8+5e^2+11e^4-24e^6)F(e,\ \tfrac{1}{2}\pi)$$

$$-(8+9e^6+16e^4-48e^6)E(e,\ \tfrac{1}{2}\pi)]\ldots\ldots\ (22).$$

$$\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}\sin^4\theta\cos^2\theta d\theta=\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta\cos^2\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}-e^2\int_0^{\frac{1}{2}\pi}\frac{\sin^6\theta\cos^2\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}$$

∴ From (11) and (18) by substitution we get,

$$\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}\sin^4\theta\cos^2\theta d\theta=\frac{1}{105e^6}[(8-5e^2-5e^4+8e^6)E(e,\ \tfrac{1}{2}\pi)$$

$$-(8-9e^2-3e^4+4e^6)F(e,\ \tfrac{1}{2}\pi)]\ldots\ldots(23).$$

$$\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}\sin^2\theta\cos^4\theta d\theta=\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^4\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}-e^2\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta\cos^4\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}$$

∴ From (12) and (19) by substitution we get,

$$\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}\sin^2\theta\cos^4\theta d\theta=\frac{1}{105e^6}[(3-23e^2+18e^4-3e^6)F(e,\ \tfrac{1}{2}\pi)$$

$$-(8-19e^2+9e^4-6e^6)E(e,\ \tfrac{1}{2}\pi)]\ \ldots\ldots\ (24).$$

$$\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}\cos^6\theta d\theta=\int_0^{\frac{1}{2}\pi}\frac{\cos^6\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}-e^2\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^6\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}.$$

∴ From (13) and (20) by substitution we get,

$$\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}\cos^6\theta d\theta=\frac{1}{105e^6}[(8-32e^2+58e^4+15e^6)E(e,\ \tfrac{1}{2}\pi)$$

$$-(8-32e^2+74e^4-45e^6)F(e,\ \tfrac{1}{2}\pi)]\ldots\ldots\ (25).$$

Let $n=3$ in (10_0), then $A_5=\frac{8}{9}.\frac{1+e^2}{e}.A_4-\frac{7}{9}A_3$.

$$\therefore\ A_5=\frac{1}{\pi}\int_0^{2\pi}\frac{\cos5\varphi d\varphi}{[1+e^2-2e\cos\varphi]^{\frac{1}{2}}}$$

$$=\frac{4}{315\pi e^5}[(128+40e^2+39e^4+44e^6+64e^8)F(e,\ \tfrac{1}{2}\pi)$$

$$-(128+104e^2+99e^4+104e^6+128e^8)E(e,\ \tfrac{1}{2}\pi)]\ldots\ldots(19_0).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^{10}\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}=\frac{1}{315e^{10}}[(128+40e^2+39e^4+44e^6+64e^8)F(e,\ \tfrac{1}{2}\pi)$$

$$-(128+104e^2+99e^4+104e^6+128e^8)E(e,\ \tfrac{1}{2}\pi)]\ \ldots\ldots(26).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^8\theta\cos^2\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}=\frac{1}{315e^{10}}[(128-40e^2-21e^4-16e^6-16e^8)E(e,\ \tfrac{1}{2}\pi)$$

$$-(128-104e^2-9e^4-7e^6-8e^8)F(e,\ \tfrac{1}{2}\pi)]\ldots\ldots(27).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^6\theta\cos^4\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}=\frac{1}{315e^{10}}[(128-248e^2+111e^4+5e^6+4e^8)F(e,\ \tfrac{1}{2}\pi)$$

$$-(128-184e^2+27e^4+11e^6+8e^8)E(e,\ \tfrac{1}{2}\pi)]\ldots\ldots\ (28).$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^4\theta\cos^6\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}} = \frac{1}{315e^{10}}[(128-328e^2+243e^4-25e^6-10e^8)E(e,\ \tfrac{1}{2}\pi)$$

$$-(128-392e^2+399e^4-130e^6-5e^8)F(e,\ \tfrac{1}{2}\pi)] \ldots\ldots (29).$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta\cos^8\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}} = \frac{1}{315e^{10}}[(128-536e^2+855e^4-622e^6+175e^8)F(e,\ \tfrac{1}{2}\pi)$$

$$-(128-472e^2+627e^4-334e^6+35e^8)E(e,\ \tfrac{1}{2}\pi)] \ldots\ldots (30).$$

$$\int_0^{\frac{1}{2}\pi} \frac{\cos^{10}\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}} = \frac{1}{315e^{10}}[(128-616e^2+1179e^4-1126e^6+553e^8)E(e,\ \tfrac{1}{2}\pi)$$

$$-(128-680e^2+1479e^4-1691e^6+1069e^8-315e^{10})F(e,\ \tfrac{1}{2}\pi)] \ldots\ldots (31).$$

$$\int_0^{\frac{1}{2}\pi} \sqrt{[1-e^2\sin^2\theta]}\sin^8\theta d\theta = \frac{1}{315e^8}[(16+8e^2+12e^4+28e^6-64e^8)F(e,\ \tfrac{1}{2}\pi)$$

$$-(16+16e^2+21e^4+40e^6-128e^8)E(e,\ \tfrac{1}{2}\pi)] \ldots\ldots (32).$$

$$\int_0^{\frac{1}{2}\pi} \sqrt{[1-e^2\sin^2\theta]}\sin^6\theta\cos^2\theta d\theta = \frac{1}{315e^8}[2(8-4e^2-3e^4-4e^6+8e^8)E(e,\ \tfrac{1}{2}\pi)$$

$$-(16-16e^2-3e^4-5e^6+8e^8)F(e,\ \tfrac{1}{2}\pi)] \ldots\ldots (33).$$

$$\int_0^{\frac{1}{2}\pi} \sqrt{[1-e^2\sin^2\theta]}\sin^4\theta\cos^4\theta d\theta = \frac{1}{315e^8}[4(4-10e^2+6e^4+e^6-e^8)F(e,\ \tfrac{1}{2}\pi)$$

$$-(16-32e^2+9e^4+7e^6-8e^8)E(e,\ \tfrac{1}{2}\pi)] \ldots\ldots (34).$$

$$\int_0^{\frac{1}{2}\pi} \sqrt{[1-e^2\sin^2\theta]}\sin^2\theta\cos^6\theta d\theta = \frac{1}{315e^8}[2(8-28e^2+33e^4-10e^6+5e^8)E(e,\ \tfrac{1}{2}\pi)$$

$$-(16-64e^2+93e^4-50e^6+5e^8)F(e,\ \tfrac{1}{2}\pi)] \ldots\ldots (35).$$

$$\int_0^{\frac{1}{2}\pi} \sqrt{[1-e^2\sin^2\theta]}\cos^8\theta d\theta = \frac{1}{315e^8}[4(4-22e^2+51e^4-68e^6+35e^8)F(e,\ \tfrac{1}{2}\pi)$$

$$-(16-80e^2+165e^4-194e^6-35e^8)E(e,\ \tfrac{1}{2}\pi)] \ldots\ldots (36).$$

Thus we can proceed as far as we desire, finding the values of similar expressions almost without number.

[To be Continued.]